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This center and the intersections are shown on the figure. Whence we read the roots $-1$, $-2$ and $3$. If the equation had been $x^3 - 700x - 6{,}000 = 0$, that is, the roots ten times as large, we would take $p = 5$ and the figure would be identical, the unit only being changed. The curve $(x - 2p)^3 = \frac{27}{2}py^2$, which gives the criterion for the nature of the roots is also shown. The roots are: one real and two imaginary, two equal or three real and unequal, according as the center $(d, e)$ is to the left of, on, or to the right of, this curve. The accuracy of the solution varies with the nature of the intersections.

# SOME PROPERTIES OF THE NORMALS TO PARABOLAS.

By S. G. BARTON, University of Pennsylvania.

There are in general three normals to a parabola from any point. If the point lies on the evolute of the parabola, two of the normals coincide (all three coincide at the cusp); if on the concave side of the evolute two normals are imaginary; if on the convex side, the three normals are real and distinct. The following properties referring chiefly to normals drawn from points on the evolute were found in an investigation for another purpose.

Consider the parabola $y^2 = 4px$, whose evolute is $27py^2 = 4(x - 2p)^3$. If $(a, b)$ is a point on the evolute, then

(1) The equations of the coincident and single normals are respectively

$$y - b = \sqrt{\frac{a - 2p}{3p}}\,(x - a) \qquad \text{and} \qquad y - b = -2\sqrt{\frac{a - 2p}{3p}}\,(x - a)$$

and coördinates of the feet of these normals are

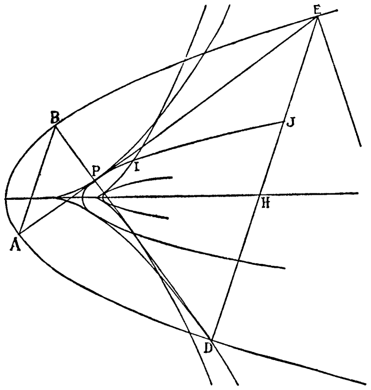
$$\left(\frac{a - 2p}{3},\ -2\sqrt{\frac{(a - 2p)p}{3}}\right) \qquad \text{and} \qquad \left(\tfrac{4}{3}(a - 2p),\ 4\sqrt{\frac{(a - 2p)p}{3}}\right).$$

Thus the abscissa of the foot of the single normal is four times that of the double normal, and the ordinate of the foot of the former is minus twice that of the latter. The abscissa of the foot of the double normal is one third of the difference between the abscissæ of the point $(a, b)$ and the cusp of the evolute, and the difference of the abscissæ of the feet is equal to the difference for the point and cusp.

(2) The $x$ intercepts are $\frac{1}{2}(a + 4p)$ and $\frac{1}{3}(4a - 2p)$. Hence the foot of the ordinate of the point $(a, b)$ trisects the distance between the intersections of the normals with the $x$ axis, being nearer the single normal. The difference of the two intercepts is $(a - 2p)$ which is the distance from the foot of the ordinate to the cusp. Thus the single normal cuts the $x$ axis four times as far from the cusp as the double normal. Similarly the foot of the abscissa trisects the distance between the intersections of the normals with the $y$ axis, being nearer the double normal. The $x$ intercept of the double normal is a mean proportional between the abscissæ of the points in which it intersects the parabola.

(3) The difference of the squares of the lengths of the normals (point to curve) equals three times the square of the difference of the abscissæ of the point and the cusp. The square on the line joining the feet of the normals is three times the square on the line joining the vertex with the foot of the double normal. The double normal (point to curve) is divided by the axis in a ratio which is twice that into which the axis divides the single normal. The locus of the intersection of two normals which are perpendicular to each other is the parabola $y^2 = p(x - 3p)$. This locus touches the evolute at $\left(\frac{1}{2}p, \pm \frac{\sqrt{2}}{2} p\right)$.

(4) These points and the normals through them are particularly interesting. The single normal through the upper of these points, $P$ in the figure, is $y = -\sqrt{2}(x - 4p)$. This line is normal to both parabolas, normal to the evolute at $P$ and tangent to the evolute at $D$.



If we draw the evolute of the parabola of perpendicular normals, $y^2 = p(x - 3p)$, which we call the second parabola, and its parabola of perpendicular normals, and so on we get a series of parabolas and evolutes each having the line $y = -\sqrt{2}(x - 4p)$ as a normal. The latus rectum of each parabola is one fourth that of the preceding and the cusp of each evolute is at the foot of the ordinate of the point in which the common normal cuts the parabola. $DB$ is the radius of curvature at $B$.

(5) The similar tangents and normals to the parabolas and evolutes at the points where the common normal cuts them are parallel. The tangent at the lower point of intersection in each case passes through the focus of the preceding parabola.

(6) The common normal and its correspondent constitute the envelope of the evolutes of the series of parabolas $y^2 = 4p'(x - 4p + 4p')$, in which $p'$ is the variable parameter. The parabolas considered are particular ones of the series.

(7) The length of the common normal is $6\sqrt{3}\,p$. It is trisected by the axis,

bisected at one point of intersection with the second parabola, and quadrisected by the other, smaller divisions being made by the other parabolas. The length of the double normal is $6\sqrt{6}\,p$. It is divided by the axis into parts which are as 1 to 5, and by the evolute into parts which are as 5 to 27 and 1 to 3.

(8) The line $AB$, joining the feet of the normals, passes through the focus (as is true for any pair of perpendicular normals), is $4\frac{1}{2}$ times $p$ in length, is trisected at the focus, is parallel to $DE$ (joining the other ends of the normals through $P$) and equal to $\frac{1}{3}DE$. Both $AB$ and $DE$ are bisected by the diameter through $P$ and are of course parallel to the tangent at its extremity.

(9) $DE$ is itself a normal to the parabola at $D$. If we draw the normal at $E$, then $DE$ bisects the angle between the other two normals through $E$, the two to the right forming with the axis an isosceles triangle.

(10) The cusp and the points $P$ and $I$ are collinear, $P$ being the point of bisection. The focus and the points $P$ and $J$ are collinear, $P$ being the point of quadrisection, and the join being perpendicular to the normal at $E$.

(11) $P$ is equidistant from $B$ and the focus. The vertex, $A$, $B$ and the middle point of $BD$ lie on a circle whose center is on $AE$. The point $I$ is equidistant from the cusp and $B$. The focus is equidistant from the intersections of the evolute with the parabola and $H$. A circle with $BD$ as a diameter passes through the vertex. $D$ and $B$ are equidistant from $J$ and $AB = JE$.

(12) The perpendicular bisector of $DH$ is tangent to the second parabola where $BD$ cuts it and the normal to the evolute at this point passes through $J$. The perpendicular to $DE$ at $H$ passes through $B$.

(13) The chord joining the vertex with the upper intersection of the parabola and evolute is perpendicular to $BD$ and trisected by it.

(14) The third normal from any point on the second parabola is trisected by the axis.

(15) The areas between the normal chords $AE$ and $BD$ and the parabola are equal. Moreover the three areas between the chords $BE$, $AB$, and $AD$ and the parabola are equal.

(16) From any point whatever $(a, b)$ the sum of the $X$ intercepts is $2(a + p)$, that is, twice the distance of the point from the directrix, and is constant for all points on a line perpendicular to the axis. Likewise the sum of the focal distances of the feet of the three normals is constant and equal to $(2a - p)$.

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# A SIMPLE ALGEBRAIC PARADOX.

By J. L. COOLIDGE, Harvard Universtiy.

Given two linear homogeneous complex equations

$$(a + bi)(p + qi) + (c + di)(r + si) = 0, \qquad (1)$$
$$(a' + b'i)(p + qi) + (c' + d'i)(r + si) = 0.$$